Complex Numbers Notes

Theta Division
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1 Introduction

1.1 What is a Complex Number?

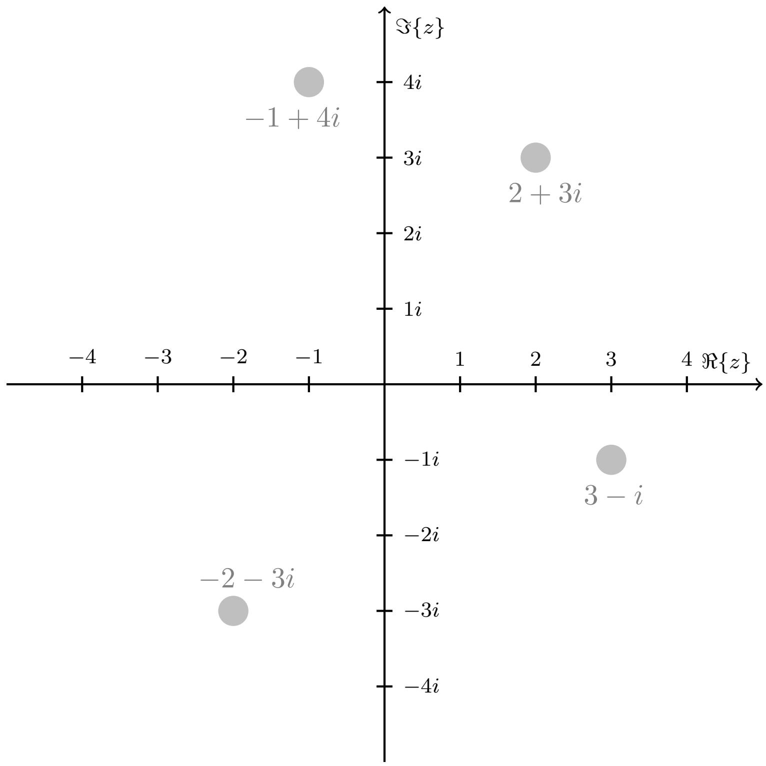
A complex number is a number of the form $a + bi$ where a and b are real numbers and $i = \sqrt{-1}$. Complex numbers are typically referenced by the letter "z".

1.2 Vocabulary

- Real Part: Typically written as $\text{Re}(a + bi)$, the real part of a complex number is a.
- Imaginary Part: Typically written as $\text{Im}(a + bi)$, the real part of a complex number is b.
- Complex Conjugate: The complex conjugate of a complex number, $a+bi$, is $a-bi$. Multiplying a complex number by it's conjugate will always result in a real number.
- Absolute Value/Magnitude/Modulus: All of these terms refer to the same thing, the distance from a complex number to the origin on the complex plane. It is typically written as $|z|$ and has a value of $\sqrt{a^2 + b^2}$

1.3 Argand Plane

The Argand plane, also known as the complex plane is a system used to graph complex numbers. Similar to the regular grid system, there are two axis, real and imaginary. The real axis is horizontal and the imaginary axis is vertical. Similar to how a coordinate (x, y) would be graphed x units along the x-axis and y units along the y-axis, a complex number $a + bi$ would be graphed a units along the real axis and b units along the imaginary axis.



$\Im\{z\}$
$4i$
$-1+4i$
$3i$
$2+3i$
$2i$
$1i$
-4
-3
-2
-1
1
2
3
4
$\Re\{z\}$
$-1i$
$3-i$
$-2i$
$-2-3i$
$-3i$
$-4i$

2 Powers of Complex Numbers

2.1 Powers of i

As $i = \sqrt{-1}$, $i^2 = -1$. Since $i^3 = i \times i^2$ it comes as no surprise that $i^3 = -i$. Furthermore, i^4 can be written as $(i^2)^2$; therefore, $i^4 = 1$.

Note 1. As i raised to any power higher than 4 can always have i^4 factored out, when finding i^x, all you need to do is find $i^{x \, mod \, 4}$!

3 Operations with Complex Numbers

3.1 Addition and Subtraction

Adding and subtracting complex numbers is a lot like adding and subtracting binomials: you group like pairs. The real and imaginary parts are grouped then added or subtracted.

$$(a+bi)+(c+di) = a+c+bi+di = (a+c)+(b+d)i \tag{1}$$

$$(a+bi)-(c+di) = a-c+bi-di = (a-c)+(b-d)i \tag{2}$$

3.2 Multiplication

Multiplication is also a lot like multiplying binomials! There are many possible methods for multiplying, but my favorite is FOIL.

$$(a+bi)\times(c+di) = ac+adi+bci+bdi^2 = ac-bd+adi+bci = (ac-bd)+(ad+bc)i \tag{3}$$

3.3 Division

Unfortunately dividing by a complex number is a bit trickier. When you have $\frac{x}{a+bi}$, you should multiply by the conjugate of the denominator ($\frac{a-bi}{a-bi}$) in order to make the denominator a real number.

$$\frac{x}{a+bi} = \frac{x}{a+bi}\times\frac{a-bi}{a-bi} = \frac{x\times(a-bi)}{a^2+b^2} \tag{4}$$

4 Polynomials with Complex Roots

4.1 Complex Conjugate Root Theorem

The complex conjugate root theorem states that when there is a polynomial with all real coefficients if we know that a complex number is one of the roots, then we know the conjugate of that complex number is also a root. For example, say we have the polynomial $x^2 - 4x + 5$. If we know that one of the roots of the polynomial is a complex number ($2 - i$ in this case), then by the complex conjugate root theorem the conjugate of that number ($2 + i$) is also a root of the polynomial. This can be verified by multiplying $(x - (2 - i))$ and $(x - (2 + i))$.

Note 2. By the complex conjugate root theorem, when a complex number is a root of a polynomial with all real coefficients, its conjugate is also a root.

4.2 Descartes' Rule of Signs

Descartes' Rule of Signs can be used to determine the possible number of real roots of a polynomial. However, it will NOT tell you what the roots are. To demonstrate, I will use the polynomial $x^4 - 2x^3 - 6x^2 + 22x - 15$.

To find the maximum number of positive real roots of the polynomial, count the number of sign changes between coefficients. In the example, it is 3. So our polynomial could have up to 3 positive real roots.

To find the maximum number of negative real roots, plug in $-x$ and count the sign changes. This would leave us with the polynomial $x^4 + 2x^3 - 6x^2 - 22x - 15$ which has 1 sign change. So we can have at most 1 negative real root.

However, remember that we can also have imaginary roots which come as pairs of 2. So we need to subtract multiples of 2 from the number of sign changes. This means that although we could have 3 positive real roots, we could also have just 1 if we have 2 imaginary roots. Since we can have a max of 1 negative real root, it is impossible to subtract a multiple of 2. This means that while we are guaranteed to have exactly 1 negative real root, we can have either 1 or 3 positive real roots.

5 Practice Problems

5.1 Problems

1. The complex number $z = 5 - 12i$. Find $|z|$.
    - Source: Unknown Year Algebra 2 January Regional Problem #12
2. Write $\dfrac{1-2i}{3+4i}$ in standard form.
    - Source: Unknown Year Algebra 2 January Regional Problem #12
3. For $i = \sqrt{-1}$, simplify $\dfrac{i^{2021} + i^{2020} + i^{2019}}{i+1}$
    - Source: 2021 Algebra 2 January Regional Problem #19
4. Evaluate $(1-i)^6$
    - Source: Unknown Year Algebra 2 January Regional Problem #2
5. Rationalize: $5i\left(\dfrac{2i-3}{2-i} - \dfrac{3i-2}{2+i}\right)$
    - Source: Unknown Year Algebra 2 January Regional Problem #19
6. $|3-i||3+i| \times \dfrac{9-i}{|2-94i|^2} = \dfrac{a+bi}{c}$. Find a+b+c if a, b, and c are positive relatively prime integers.
    - Source: Unknown Year Algebra 2 January Regional Problem #23
7. Evaluate $\dfrac{9-i}{3+i}\left(\dfrac{1+i}{3+i} - \dfrac{i-11}{2-i}\right)$.
    - Source: Unknown Year Algebra 2 January Regional Problem #25

5.2 Solutions

1. $|5-12i| = \sqrt{5^2+12^2} = 13$
2. $\dfrac{1-2i}{3+4i} = \dfrac{1-2i}{3+4i} \times \dfrac{3-4i}{3-4i} = \dfrac{-5-10i}{25} = \dfrac{-1-2i}{5}$
3. $\dfrac{i^{2021} + i^{2020} + i^{2019}}{i+1} = \dfrac{i^1 + i^0 + i^3}{1+i} = \dfrac{i+1-i}{1+i} = \dfrac{1}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{1-i}{2}$
4. $(1-i)^2 = -2i$
   $(1-i)^3 = (1-i)^2 \times (1-i) = -2i \times 1 - i = -2 - 2i$
   $(1-i)^6 = ((1-i)^3)^2 = (-2-2i)^2 = -8i$
   A much cleaner way of solving this problem is to use Euler's Formula. However that is beyond the scope of this handout.

5. $5i\left(\frac{2i-3}{2-i}-\frac{3i-2}{2+i}\right)=5i\left(\frac{2i-3}{2-i}\times\frac{2+i}{2+i}-\frac{3i-2}{2+i}\times\frac{2-i}{2-i}\right)=5i\left(\frac{-8+i}{5}-\frac{-1+8i}{5}\right)$
$5i\left(\frac{-7-7i}{5}\right)=7-7i$

6. $|3-i||3+i|\times\frac{9-i}{|2-94i|^2}=\sqrt{10}\times\sqrt{10}\times\frac{9-i}{2^2+(-94)^2}=10\times\frac{9-i}{8840}=\frac{9-i}{884}$.
$a+b+c=9-1+884=892$

7. $\frac{9-i}{3+i}\left(\frac{1+i}{3+i}-\frac{i-11}{2-i}\right)=\frac{9-i}{3+i}\times\frac{3-i}{3-i}\left(\frac{1+i}{3+i}\times\frac{3-i}{3-i}-\frac{i-11}{2-i}\times\frac{2+i}{2+i}\right)$
$=\frac{26-12i}{10}\left(\frac{4+2i}{10}-\frac{-23-9i}{5}\right)=\frac{13-6i}{5}\left(\frac{2+i}{5}-\frac{-23-9i}{5}\right)$
$=\frac{13-6i}{5}\left(\frac{25+10i}{5}\right)=\frac{13-6i}{5}(5+2i)=\frac{(13-6i)(5+2i)}{5}=\frac{77-4i}{5}$