2021 - 2022 Student Delegates Invitational

Problems by
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Theta Individual

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1. Find $f(2)$ if $f(x) = x^3 - 7x^2 + 16x - 2034$.

   A. -2020 B. -2021 C. -2022 D. -2023 E. NOTA

2. Find a in terms of m_1,m_2, and g

$$\begin{cases} T - m_1 g = m_1 a \\ T - m_2 g = -m_2 a \end{cases}.$$

   A. $\frac{m_2 - m_1}{m_1 + m_2} g$ B. $\frac{m_1 + m_2}{m_2 - m_1} g$ C. $\frac{m_1}{m_2} g$ D. $\frac{m_2}{m_1} g$ E. NOTA

3. The length of Corbin's hair can be modeled by the function $L(t) = 1/(t^2 - 34t + 290)$ where the length is in meters and time is in years. What is the maximum length of Corbin's hair?

   A. $\frac{1}{290}$ m B. $\frac{1}{17}$ m C. 1 m D. No maximum E. NOTA

4. How many asymptotes does $LILA(x) = \dfrac{x^2 + 2x + 1}{x^2 + 4x + 3}$ have?

   A. 0 B. 1 C. 2 D. 3 E. NOTA

5. Its Mr. Rovere's Birthday! To celebrate he is wearing a conical hat with a diameter of 2 cm and a height of 3 cm. What is the volume (in cm^3) of his hat to the nearest integer?

   A. 3 B. 6 C. 13 D. 25 E. NOTA

6. It takes 4 Thetas exactly 4 hours to fail 4 tests. How many hours does it take 1 Theta to fail 1 test?

   A. 4 B. 3 C. 2 D. 1 E. NOTA

7. How many subsets of a 10 element set are there such that each subset has an even number of elements?

   A. 0 B. 256 C. 511 D. 512 E. NOTA

8. Find the sum of the inradius and circumradius of an equilateral triangle of side length $4s$.

   A. $s\sqrt{3}$ B. $2s\sqrt{3}$ C. $4s\sqrt{3}$ D. $6s\sqrt{3}$ E. NOTA

9. Convert the base 10 decimal 0.625 to base 2.

   A. 0.110_2 B. 0.101_2 C. 0.100_2 D. 0.111_2 E. NOTA

10. Tiger has skipped math practice to fish! At 12PM, he drops his anchor into 85 feet deep water, with his line having no slack. At 3PM, the water level drops 1 foot. With this additional slack, what is the area of the region Tiger's boat can drift into? (Ignore the dimensions of the boat)

A. 13π B. 225π C. 15π D. 169π E. NOTA

11. If Anastasia, Akshay, Sebastian, Guilherme, and 3 Dr. Santos clones (you cannot tell them apart unfortunately) are sitting at a special circular table with 7 seats, how many seating arrangements exist where Anastasia and Akshay sit adjacent to each other?

A. 40 B. 120 C. 240 D. 480 E. NOTA

12. If the roots of $p(x) = x^3 + 5x^2 + 7x + 10$ are r, s and t, what is the value of $\frac{2}{r^2} + \frac{2}{s^2} + \frac{2}{t^2}$?

A. $\frac{51}{50}$ B. $-\frac{51}{100}$ C. $\frac{51}{100}$ D. $-\frac{51}{50}$ E. NOTA

13. Find the maximum area of a rectangle that can be inscribed in an equilateral triangle of side length 3.

A. $\frac{9\sqrt{3}}{16}$ B. $\frac{9\sqrt{3}}{4}$ C. $\frac{3\sqrt{3}}{8}$ D. $\frac{9\sqrt{3}}{8}$ E. NOTA

14. Find the product of the roots of $5^{84} * x^{\log_5 x\sqrt{x}} = x^{24}$.

A. $\frac{1}{16}$ B. 16 C. $\frac{1}{56}$ D. 56 E. NOTA

15. If $g(x) = \dfrac{x^3 - x^2 - 3x - 9}{x^2 - 6x + 8}$, then find $\lfloor g(2022) \rfloor$.

A. 2025 B. 2026 C. 2027 D. 2028 E. NOTA

16. Let N be the number of possible rational roots of $100x^3 - 69x^2 + 420x - 144$ according to the rational root theorem. Find N.

A. 54 B. 63 C. 108 D. 126 E. NOTA

17. A circle with center V with radius 2 rolls along the triangle $\triangle YOM$ which has side lengths 14, 48, and 50 always remaining in contact with at least one side of the triangle. After V returns to its starting position for the third time, how many revolutions has a point on the circle A travelled?

A. 224 B. $\frac{224}{3}$ C. $\frac{56}{\pi}$ D. $\frac{56}{3\pi}$ E. NOTA

18. Solve $3 \cdot 15^x + 5^{2x+1} - 3 \cdot 3^{2x} = 0$

A. $\dfrac{\log 3 - \log 5}{\log 2 - \log 5}$ B. $\dfrac{\log 3 - \log 5}{\log 5 - \log 2}$ C. $\dfrac{\log 2 - \log 5}{\log 5 - \log 3}$ D. $\dfrac{\log 2 - \log 5}{\log 3 - \log 5}$ E. NOTA

19. As you stay in the *Monty Hall Hotel* ^{TM}, Jack$^{\text{f}}$ wants to play a game with you... yes, you! Jack$^{\text{f}}$ has 4 doors, and behind each of them is a player: Rajam, Himashi, Kangana, and Samuil. Himashi, Kangana, and Samuil have nothing to offer you, since they spent all their money on food for the night, although Rajam has \$5000 for you if you pick their door. Your goal is to acquire the largest amount of money possible. Jack$^{\text{f}}$ asks you to pick a door, and then opens an adjacent door to reveal that Himashi is behind it! He then asks you if you would like to switch doors... What is the positive difference in the expected amount of money that you would receive if you switched doors as opposed to not switching doors?

A. \$625 B. \$1250 C. \$2500 D. \$0 E. NOTA

20. Consider the sequence a_n defined recursively as

$$a_n = \begin{cases} 1 & n = 0 \\ 2022a_{n-1} + 1 & n > 0 \end{cases}$$

Find the value of a_{2021}.

A. $\dfrac{2022^{2021} - 1}{2021}$ B. $\dfrac{2022^{2023} - 1}{2021}$ C. $\dfrac{2022^{2022} - 1}{2021}$ D. $\dfrac{2021^{2022} - 1}{2021}$ E. NOTA

21. Define the *index* of a rectangle as the rectangle's area plus its perimeter. Furthermore, define the *rectangle space* of a number as the number of non-congruent rectangles that have that number as its index. For example, the rectangle space of 20 is 2 because there are only 2 rectangles with an index of 20 (1×6 and 2×4). Find the smallest possible number with a rectangle space of 10.

A. 572 B. 476 C. 356 D. 236 E. NOTA

22. Find the sum of the abscissas and ordinates of the coordinates of all the intersection points between the conics $2x^2 - 5y^2 + 16x + 60y - 158 = 0$ and $7x^2 + 13y^2 + 56x - 156y + 489 = 0$. (If these conics do not intersect, select E. NOTA).

A. -8 B. -4 C. 4 D. 8 E. NOTA

23. Rolando and Hanna are in a epic duel to see who can calculate the following sum for the largest n:

$$\frac{1}{3n^3} + \frac{4}{3n^3} + \frac{9}{3n^3} + \cdots + \frac{n^2}{3n^3}$$

Prove to them whose boss by finding the limit of the sum as n approaches infinity.

A. $\dfrac{1}{9}$ B. $\dfrac{1}{3}$ C. $\dfrac{2}{3}$ D. $\dfrac{1}{2}$ E. NOTA

24. The *population standard deviation* of a set of numbers $\{x_1, x_2, ... x_n\}$ is given by the following formula:

$$\sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n}}$$

where $\bar{x}$ denotes the arithmetic mean of the set of numbers. Find the population standard deviation of the roots of the polynomial $P(x) = x^4 - 6x^3 + 8x^2 + 2x - 1$, which has only real roots.

A. $\frac{\sqrt{11}}{2}$ B. $\frac{\sqrt{19}}{2}$ C. $\frac{\sqrt{29}}{2}$ D. $\frac{\sqrt{37}}{2}$ E. NOTA

25. An non-rotated ellipse whose eccentricity is $\frac{1}{2}$ is centered at the origin and includes the point $(2, 2)$. Find the area enclosed by the ellipse.

A. $\frac{14\pi\sqrt{3}}{3}$ B. $\frac{7\pi\sqrt{3}}{3}$ C. $\frac{2\pi\sqrt{21}}{3}$ D. $\frac{\pi\sqrt{21}}{3}$ E. NOTA

26. Since he forgot to study and has nothing better to do, Apollo is trying to calculate his odds of getting an A+ from pure guessing on his Spanish test. The test is made up of 10 fill-in-the-blank questions. Luckily, the teacher provided a word bank containing 15 words. Every answer to a fill-in-the-blank question must come from the word bank, and each word in the word bank can be used once, more than once, or not at all. Before he decides to guess the correct order, he first agrees upon a set of 10 answer choices (not necessarily distinct, since answers can be used more than once) to use so he can narrow down his options. What is the probability that this set is correct?

A. $\frac{10! \cdot 15!}{25!}$ B. $\frac{10! \cdot 14!}{24!}$ C. $\frac{10! \cdot 5!}{15!}$ D. $\frac{10! \cdot 4!}{14!}$ E. NOTA

27. Find the minimum value of $M(x) = \sqrt{x^2 - 10x + 26} + \sqrt{x^2 - 18x + 85}$

A. 0 B. 3 C. $2\sqrt{2} + \sqrt{5}$ D. 5 E. NOTA

28. Bored out of their mind, Sahir selects non-zero real numbers a, b and c and suddenly realizes they hold the following property:

$$bc + \frac{13}{a} = ac + \frac{12}{b} = ab + \frac{6}{c} = \frac{1}{a+b+c}$$

What a coincidence! Find $a + b + c$.

A. $-\frac{1}{2}\sqrt[3]{\frac{10}{3}}$ B. $4\sqrt[3]{\frac{10}{3}}$ C. $-4\sqrt[3]{\frac{10}{3}}$ D. $\frac{1}{2}\sqrt[3]{\frac{10}{3}}$ E. NOTA

29. The hyperbola with equation $x^2 - xy + 2x - 5y + 3 = 0$ has a center at (h, k). Find $h + k$.
(Hint: you do not need to rotate to find this value)

A. 5 B. 7 C. -13 D. -15 E. NOTA

30. Corbin taught the Thetas how do to this question, so lets see if they paid attention. Let $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n > 1$. Evaluate

$$\sum_{n=0}^{\infty} \frac{F_n}{5^n}$$

A. $\frac{25}{24}$ B. $\frac{25}{19}$ C. $\frac{5}{3}$ D. $\frac{5}{4}$ E. NOTA