2022 Student Delegates Invitational

Problems by
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Mu Individual

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1. Welcome! To kick off the test on the 22nd day of the 2022nd year, determine the following limit.

$$\lim_{x\to\infty} \frac{\sqrt{x+2022}}{\sqrt{x}+\sqrt{x+2022}}$$

A. 1 B. 1/2022 C. 1/2 D. 2 E. NOTA

2. Evaluate the derivative of $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$ at $x = 1$.

A. -5 B. 0 C. 15 D. 100 E. NOTA

3. Evaluate

$$\lim_{x\to 0} \frac{2^{2+x} - 2^{2-x}}{x}$$

A. $\ln 4$ B. $\ln 16$ C. $\ln 64$ D. $\ln 256$ E. NOTA

4. Oh no, Legosi has fallen off the Shishigumi tower! If his y-coordinate (in meters off the ground) at time $t \geq 0$ (in seconds) is given by $y(t) = -5t^2 - 2t + 7$. Compute the speed at which Legosi is falling (in meters per second) when he hits the ground.

A. 1 B. 7 C. 10 D. 12 E. NOTA

5. Evaluate $\displaystyle\int_0^1 \frac{x^{2021}}{x^{2022}+1}\,dx$.

A. $\ln(2)$ B. $\frac{\ln(2)}{2021}$ C. $\frac{\ln(2)}{2022}$ D. $\frac{\ln(2)}{2023}$ E. NOTA

6. Find the area bounded by the functions $f(x) = x^n$ and $g(x) = x^{n+1}$ in the first quadrant for non-negative n.

A. $\frac{1}{n^2+1}$ B. $\frac{1}{n^2+n}$ C. $\frac{1}{n^2+2n+1}$ D. $\frac{1}{n^2+3n+2}$ E. NOTA

7. Find the volume when the region bounded by the functions $f(x) = x^n$ and $g(x) = x^{n+1}$ in the first quadrant is rotated around the x-axis.

A. $\pi \int_0^1 x^{2n}(x^2-1)\,dx$
B. $\pi \int_0^1 x^{2n}(1-x)^2\,dx$
C. $2\pi \int_0^1 x^{2n}(1-x^2)\,dx$
D. $2\pi \int_0^1 x^{2n}(1-x)^2\,dx$
E. NOTA

8. Determine the value of the following limit.

$$\lim_{n\to\infty}\left(\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)^n$$

A. $\frac{1}{\sqrt{e}}$ B. $\frac{1}{2\sqrt{e}}$ C. $\frac{1}{e}$ D. $\frac{1}{2e}$ E. NOTA

9. To celebrate the 22nd day of the year 2022, Taylor Swift creates the following sum:

$$1+22+\frac{(22)^2}{2}+\frac{(22)^3}{6}+\frac{(22)^4}{24}+\frac{(22)^5}{120}+...+\frac{(22)^{22!}}{(22!)!}$$

When asked if she knew the value of the sum, Taylor's face grew Red and she said "I Almost Do." Help Taylor find her State of Grace by finding the value closest to Taylor's series from the following:

A. e^{22} B. $22!$ C. $e^{22!}$ D. $22!e$ E. NOTA

10. For a given rectangle, the length of one side is 16 times the length of the other. If the area is increasing at a rate of 80 in^2/min, at what rate is the longer side increasing when the area is 144?

A. $\frac{16}{3}$ B. $\frac{40}{3}$ C. $\frac{5}{3}$ D. $\frac{5}{24}$ E. NOTA

11. Find y' at the point $(\pi/3, 0)$ on the implicit graph $2\sqrt{\cos x}+2x\sin y=\sqrt{2}$

A. $-\frac{2\pi\sqrt{6}}{9}$ B. $\frac{2\pi\sqrt{6}}{9}$ C. $-\frac{3\sqrt{6}}{4\pi}$ D. $\frac{3\sqrt{6}}{4\pi}$ E. NOTA

12. Wait! Stay Stay Stay! It's not over, Babe. Come Back... Be Here! Taylor Swift has returned with more series, and this time, Everything Has Changed! If you don't solve this, We Are Never Ever Getting Back Together. I know, you're probably thinking "I Knew You Were Trouble!" but consider yourself The Lucky One. Now go, Run. Prevent the Forever Winter by becoming the Better Man and finding the interval of convergence of the following sum, where r is a real variable and o and a are positive integer constants:

$$\sum_{n=1}^{\infty}\left(\frac{(r-o)^3}{a}\right)^n\frac{1}{n}$$

A. $(o-\sqrt[3]{a}, o+\sqrt[3]{a})$ B. $[o-\sqrt[3]{a}, o+\sqrt[3]{a})$ C. $(o-\sqrt[3]{a}, o+\sqrt[3]{a}]$ D. $[o-\sqrt[3]{a}, o+\sqrt[3]{a}]$ E. NOTA

13. Evaluate the following limit.

$$\lim_{x\to 0}\left(1+2x+3x^2+4x^3+\cdots\right)^{1/x}$$

A. 1 B. $e^{\frac{1}{e}}$ C. e D. e^2 E. NOTA

14. Nico, Christina, and Soha are all trying to cram practice in before their *Mu Area and Volume* test, which they all definitely remembered they were taking.

    Each of them are calculating the volume of a solid with a base in the shape of the region closed between the functions $f(x) = x^3 - 3x^2 + x + 5$ and $g(x) = x + 1$.

    However, each of their solids have differently shaped cross sections perpendicular to the x-axis, each with a side length on the base. Nico has equilateral triangle cross sections, Christina has hexagonal cross sections, and Soha has square cross sections. If the volumes of Nico, Christina, and Soha's solids are N, C, and S respectively, find $(N + C)/S$.

    A. $\frac{3\sqrt{3}}{4}$ B. $\sqrt{3}$ C. $\frac{7\sqrt{3}}{4}$ D. $2\sqrt{3}$ E. NOTA

15. It's Dev's birthday! To celebrate, he is visiting *Yash Towers* ^{TM}. Dev is walking towards a tower that is $\frac{1}{4}$ miles tall and 2 miles away at a rate of 2 miles per hour. The maximum height he can see is 60° above the horizontal ground. What is the rate of change of the proportion of his view that is blocked by the super-rich-and-loaded guy's tower? Assume he has negligible height (lol) and no other buildings are blocking his field of vision.

    A. $\frac{\sqrt{3}}{16}$ B. $\frac{\sqrt{3}}{8}$ C. $\frac{\sqrt{3}}{4}$ D. $\frac{\sqrt{3}}{2}$ E. NOTA

16. Saathvik is trying to find a real solution to $x^2 + 1 = 0$ (unfortunately for him, it does not have any). To approximate this solution, Saathvik starts with a guess of x_0 and performs two iterations of Newton's method to get a new guess x_2. When he attempts to perform another iteration of Newton's method, it fails, as the tangent line at $x = x_2$ is horizontal! The largest possible value of x_0 can be written as $a + \sqrt{b}$ for integers a and b. Compute $a + b$.

    A. 3 B. 1 C. 2 D. 4 E. NOTA

17. Suppose f is a rational function (a function of the form $f(x) = \frac{p(x)}{q(x)}$ for polynomials p and q) such that

    $$\lim_{x \to 20} f(x) = \infty \text{ and } \lim_{x \to 22} f(x) = -\infty.$$

    Find the smallest possible value of the sum of the degrees of p and q.

    A. 3 B. 4 C. 5 D. 6 E. NOTA

18. The probability that Sharvaa will come ice skating can be modeled by the following probability density function where t is the time in days since he was last allowed to hang out:

    $$\begin{cases} P(t) = 0 & t \leq 0 \\ P(t) = \frac{2}{\sqrt{\pi}} e^{-t^2} & t > 0 \end{cases}.$$

    Find the expected number of days until he comes ice skating.

    A. 0 B. $\frac{1}{\sqrt{\pi}}$ C. $\frac{1}{2\sqrt{\pi}}$ D. $\frac{2}{\sqrt{\pi}}$ E. NOTA

19. Find the length of the parametric curve $x = t - \sin(t)$, $y = 1 - \cos(t)$ from $t = 0$ to $t = 2\pi$

A. $2\sqrt{2}$ B. 4 C. $6\sqrt{2}$ D. 8 E. NOTA

20. Find the largest value of $\cos^2\theta$, where θ is the angle between the vectors $\vec{v} = \langle 2, -1, 3\rangle$ and $\vec{u} = \langle 1, 1, t\rangle$, for some $t \in \mathbb{R}$.

A. 1 B. $\frac{9}{14}$ C. $\frac{289}{532}$ D. $\frac{19}{28}$ E. NOTA

21. Jake, who believes safe driving is just less fun, is adding large rocket boosters to his car. These rockets push the car forward with a constant force 1000 kg·m/s^2. However, as the rocket is used, fuel is used up in the car, causing it to lose mass and go faster. Given that Jake's car has mass of 1500 kg with a full tank of fuel and 1200 kg with an empty tank, and the fuel is used at a rate of 5 kg/s, what is Jake's velocity (in m/s) as soon as all the fuel is used assuming there is zero friction or air resistance? (*Hint: Force is equal to the total mass times the acceleration*)

A. $200\ln\frac{5}{4}$ B. $1000\ln\frac{5}{4}$ C. $400\ln 2$ D. $2000\ln 2$ E. NOTA

22. Evaluate the following integral.

$$\int_0^\pi \frac{2x\sin(x)}{3+\cos(2x)}\,dx$$

A. π^2 B. $\pi^2/2$ C. $\pi^2/4$ D. $\pi^2/8$ E. NOTA

23. For The Last Time, Taylor is tailoring a Taylor series to match Taylor's new series. She was given that for all $-\frac{1}{4} < x < \frac{1}{4}$,

$$\sum_{n=0}^{\infty} f(n)x^n = (1-4x)^{-1/2}$$

for some real function $f(n)$. However, she needs to compute

$$T = \sum_{n=0}^{\infty} \frac{nf(n)}{5^n}$$

instead. At first Taylor thought this was Treacherous, like it was a Sad Beautiful Tragic, but she quickly realized it was Nothing New. She knew it All too Well! That was The Moment I Knew this Girl at Home would have The Very First Night in Holy Ground knowing she was at peace. Tonight, I Bet You Think About Me in the Starlight too, but what you really need to think about, is how did Taylor solve it? Compute the sum T mentioned above.

A. $2\sqrt{5}$ B. $10\sqrt{5}$ C. $\frac{2\sqrt{5}}{5}$ D. $\sqrt{5}$ E. NOTA

24. Find the maximum possible value of $\displaystyle\int_0^\infty \frac{\ln(x)}{x^2+a^2}\,dx$, where a is a positive real number.

A. $\frac{\pi}{e}$ B. $\frac{\pi}{2e}$ C. $\frac{\pi}{4e}$ D. $\frac{\pi}{8e}$ E. NOTA

25. There exist two lines that are tangent to both the graphs of $y = x^2 + 20x + 22$ and $y = -x^2 + 20x - 22$. Compute the positive difference between the slopes of these two lines.

A. $4\sqrt{5}$ B. $2\sqrt{22}$ C. $8\sqrt{5}$ D. $4\sqrt{22}$ E. NOTA

26. Given that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$, compute $\int_{-\infty}^{\infty} x^4 e^{-x^2}\,dx$.

A. $\frac{3\sqrt{\pi}}{4}$ B. $\frac{\sqrt{\pi}}{2}$ C. $\frac{\sqrt{\pi}}{4}$ D. $\frac{3\sqrt{\pi}}{2}$ E. NOTA

27. Define $f(x)$ by the following definition.

$$f(x) = \begin{cases} 2021 & \text{if } \lfloor x \rfloor \text{ is odd} \\ 2020 & \text{if } \lfloor x \rfloor \text{ is even} \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. Determine the exact value of the series below.

$$\sum_{n=1}^{\infty} \frac{f(2^n \cdot \pi)}{2^n}$$

A. $2017 + \pi$ B. $2018 + \pi$ C. $2019 + \pi$ D. $2020 + \pi$ E. NOTA

28. Brighten and Himal are texting, excited about the upcoming Student Delegates Invitational! Himal loves to use her exclamation points, so much so that anytime Himal presses the button, it produces two '!!'. When typing out a limit she thought of, she forgot about this issue, and then realizes the mistake when she sees her message. Brighten, however, is unfazed and responds with a correct answer a few minutes later. What is the answer?

$$\lim_{n\to\infty} \frac{(2n-1)!!\sqrt{n}}{(2n)!!}$$

The double factorial $n!!$ is defined as $n!! = n \cdot (n-2) \cdot (n-4) \cdots$, where the terminating value is 1 is n is odd, and 2 if n is even.

A. $\sqrt{\pi}$ B. $\frac{1}{\sqrt{\pi}}$ C. $\frac{1}{2\sqrt{\pi}}$ D. $\frac{1}{4\sqrt{\pi}}$ E. NOTA

29. Given a real number $x \in [0, 1)$ with *binary* expansion $x = (0.b_1b_2b_3\cdots)_2$, let $f(x) = (0.b_1b_2b_3\cdots)_{10}$ to be the number obtained when re-evaluating the binary expansion of x as a decimal expansion. Two examples are shown below.

$$f(1/2) = f((0.1000\cdots)_2) = f((0.1000\cdots)_{10}) = 1/10$$
$$f(3/8) = f((0.0110\cdots)_2) = f((0.011\cdots)_{10}) = 11/1000$$

Evaluate the following integral.

$$\int_0^1 f(x)\ \text{dx}$$

A. 1/30 B. 1/26 C. 1/22 D. 1/18 E. NOTA

30. Corbin and Ritvik have entrusted Odin to maintain communication across FAMATLandia. Odin recently got his first paycheck from his job as the CEO of FAMATChat, the latest and hottest communication software. His paycheck is in the form of a sequence of coins $C_1, C_2, \cdots, C_n$ such that coin C_n comes up heads with probability $\frac{1}{n}$. Let p_n be the probability of getting an *even* number of heads if coin C_n is flipped n times. Determine the following limit.

$$\lim_{n\to\infty} p_n$$

A. $\frac{1+e^{-2}}{e^2}$ B. $\frac{1+e^{-2}}{e}$ C. $\frac{1+e^{-2}}{2}$ D. $\frac{1+e^{-2}}{4}$ E. NOTA