2022 Student Delegates Invitational

Problems by
Ritvik Teegavarapu, Corbin Diaz, Anagh Sangavarapu, Jacob Buchsbaum

Alpha Individual

January 22, 2022

1. Which of the following angles could correspond to $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ on the unit circle?

A. $-\dfrac{\pi}{6}$ B. $-\dfrac{2\pi}{3}$ C. $-\dfrac{5\pi}{6}$ D. $\dfrac{7\pi}{6}$ E. NOTA

2. Find $\arcsin(\cos(\arctan(-\sqrt{3})))$.

A. $\dfrac{\pi}{6}$ B. $\dfrac{5\pi}{6}$ C. $-\dfrac{\pi}{6}$ D. $\dfrac{7\pi}{6}$ E. NOTA

3. The height of Anik's vertical jump can be modelled by $f(x) = \big|17\cos(3x - \pi)\big| - 12$. What is the sum of the period, amplitude, phase shift, and vertical shift of $f(x)$?

A. $\pi + 5$ B. $\frac{2\pi}{3} + 5$ C. $\pi - \frac{7}{2}$ D. $\frac{2\pi}{3} - \frac{7}{2}$ E. NOTA

4. Which of the following is equivalent to $\dfrac{2\tan\theta}{1+\tan^2\theta}$?

A. $2\sin\theta$ B. $2\cos\theta$ C. $\sin 2\theta$ D. $\cos 2\theta$ E. NOTA

5. If $\cos\theta = \frac{1}{2}$ and $\sin\phi = \frac{1}{3}$, where $0 < \theta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \phi < \pi$, find $\cos(\theta + \phi)$.

A. $\dfrac{2\sqrt{2}+\sqrt{3}}{6}$ B. $-\dfrac{2\sqrt{2}+\sqrt{3}}{6}$ C. $\dfrac{\sqrt{3}-2\sqrt{2}}{6}$ D. $\dfrac{2\sqrt{2}-\sqrt{3}}{6}$ E. NOTA

6. Let z and w be complex numbers such that $Arg(z) = \theta$ and $Arg(w) = \phi$, where $\theta, \phi \in (0, \frac{\pi}{2})$. Which of the following complex numbers has an argument of $\theta + 3\phi$?

A. $z(3w)$ B. $z + 3w$ C. zw^3 D. $e^{i(z+3w)}$ E. NOTA

7. Romir's eyes can be modeled by the polar equation $r^2 = 4\cos 2\theta$. Find the product of their total width along the x-axis and their total area.

A. 4 B. 16 C. 8 D. $4\sqrt{2}$ E. NOTA

8. Let $\vec{r} = 2022\hat{\imath} + 2021\hat{\jmath} + 2020\hat{k}$ and $\vec{s} = 4\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$. Find $(\vec{r} \times \vec{s}) \cdot (12\vec{r} - 7\vec{s})$.

A. 2021 B. 2022 C. 2023 D. 2024 E. NOTA

9. Let S be the set of triangles $\triangle MEG$ such that $\angle M = 30°$, $EG = 6$, and $ME = 12$, and let the size of this set be X. Consider another triangle $\triangle HAN$ with $HA = 6$, $AN = X$, and $\sin(\angle A) = \frac{X}{4}$. What is the area of $\triangle HAN$?

A. $\frac{3}{4}$ B. $\frac{3}{2}$ C. 3 D. 6 E. NOTA

10. How many asymptotes exist for the function $f(x) = \dfrac{e^{2x} - 2}{3e^{2x} - 7}$?

A. 1 B. 2 C. 3 D. 4 E. NOTA

11. How many solutions exist for $5\sin x - 3\cos x + 3 = 0$ in the interval $[-2022\pi, 2022\pi)$?

A. 4043 B. 4044 C. 4045 D. 4042 E. NOTA

12. Consider the triangle ΔTIM where $TI = 14$, $IM = 15$, and $MT = 13$. There exists squares on sides TI and MT such that the base of the square lies exactly on the side of the triangle and the squares do not intersect. Let points A and B on the squaers be such that $\angle BTI = \angle ATM = 90°$. Find AB.

A. 15 B. $\sqrt{505}$ C. $4\sqrt{37}$ D. $\sqrt{673}$ E. NOTA

13. Let u be a complex number of the form $a + bi$, where $i = \sqrt{-1}$ and a and b are positive integers. Determine the sum of $a + b$ given the following.

$$(2 + i)^2(u)^2 = 50i$$

A. 4 B. 5 C. 6 D. 7 E. NOTA

14. Which of the following is never true for two matrices A and B?

A. $(AB)^{-1} = B^{-1}A^{-1}$
B. $(A + B)^2 = A^2 + B^2 + 2AB$
C. $tr(B^TA) = A \cdot B$
D. $(A + B)^T = A^T + B^T$
E. NOTA

15. Joseph completely forgot what an eigenvector is (oh Joseph!) and asks William to help him find an eigenvector to the following matrix:

$$M = \begin{bmatrix} 2 & 1-i \\ 1+i & 3 \end{bmatrix}$$

However, William also completely forgot what an eigenvector is (oh William!) and gives a vector that is **not** an eigenvector of M. Which of the following could be a vector William gave?

A. $< 1 - i, -1 >$ B. $< 1, 1 + i >$ C. $< -1, \frac{1}{2} + \frac{i}{2} >$ D. $< \frac{i-1}{4}, \frac{1}{2} >$ E. NOTA

16. The complex numbers $1-3i$ and $i+3$ are graphed in the Argand plane. What is the length of the minor axis of the locus of points for which the sum of the distances to each of the two mentioned points is 12?

A. $2\sqrt{2}$ B. $2\sqrt{5}$ C. 4 D. 8 E. NOTA

17. If $c^2-c+1=0$, what is the value of $S(c)$ for $S(x)=\sum_{n=1}^{2022} x^n$?

A. $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ B. 0 C. $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ D. 1 E. NOTA

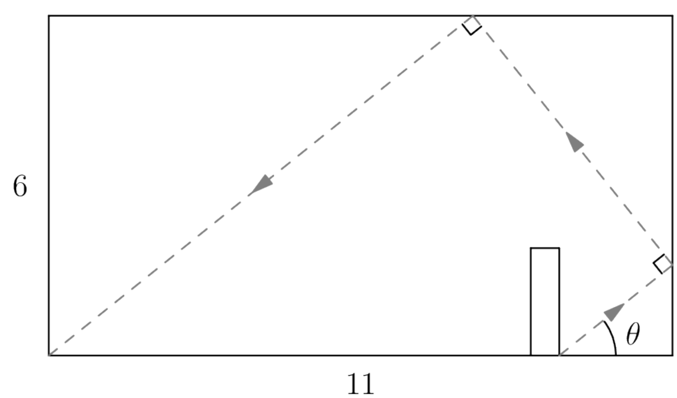
18. What is the name of the polar graph $2r=10+3\cos\theta+3\sqrt{3}\sin\theta$?

A. Cardioid B. Non-Cardiod Limaçon C. Three-Petaled Rose D. Sinusoidal Limaçon E. NOTA

19. The trajectory of Aaron's baseball hit can be modelled by the line $l(t)=\langle 2,-2,4\rangle t+\langle 2,4,0\rangle$. What is the shortest distance between $l(t)$ and and the point $(1,1,1)$?

A. $\frac{2\sqrt{6}}{3}$ B. $\frac{5\sqrt{3}}{3}$ C. $\frac{2\sqrt{3}}{3}$ D. $\frac{5\sqrt{6}}{3}$ E. NOTA

20. Carlos and Nick are playing an epic game of *Laser Tag*^{TM}. The room they are in has 2 parallel walls 6 meters tall and 11 meters apart. Nick stands against the leftmost wall. Carlos is 1 foot from the rightmost wall, and directly behind him lies a wall 1.9 meters tall that blocks his direct shot at Nick. Carlos therefore comes up with a plan: he needs to fire his laser in such a way that it first bounces off the rightmost wall then off the ceiling before directly hitting Nick. Assume the walls and ceiling are 100% reflective and the laser will always reflect in such a way that it creates a 90 degree angle (as shown below, not to scale). Further assume that Carlos and Nick can be treated like points on the floor. Let θ be an angle off the ground that Carlos can fire his laser at to successfully hit Nick as described. Find the sum of all possible values of $\tan\theta$.



A. 1 B. 2 C. 5 D. 6 E. NOTA

21. In a town, there are four points A, B, C and D. There are 0 ways to travel from point A to B, 2 from A to C, 1 from A to D, 0 from B to C, 2 from B to D, and 0 from C to D. There is 1 way to stay at each point. All routes can be taken both ways. Each journey from one point to another is considered a move. Suppose Sai starts at point A and Yusuf starts at point B. The probability that after 3 moves Sai and Yusuf are on the same point can be written as a fraction $\frac{m}{n}$ where m and n are relatively prime. What is $m + n$?

A. 218 B. 228 C. 238 D. 248 E. NOTA

22. Let $S = (0, 0)$, $Y_1 = (2, 1)$, $D = (-2, 5)$, $N = (4, 3)$, $E = (-4, 2)$, and $Y_2 = (3, 6)$. Vishakha decides to apply a transformation to all of the points. If (x, y) were the coordinates of the point originally, then (x', y') are the new coordinates, where:

$$\begin{bmatrix} 2022 & 2023 \\ 2021 & 2022 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Let H be the convex hexagon made out of these new points. Find the area of H.

A. 1 B. 17 C. 30 D. 34 E. NOTA

23. A parabola shares a latus rectum with the conic $(4\cos(\theta) + 5)r = 9$. Find the smallest distance possible between the vertex of one of these parabolas and a vertex of the other conic.

A. $\dfrac{19}{10}$ B. $\dfrac{1}{10}$ C. $\dfrac{9}{10}$ D. 1 E. NOTA

24. Find the sum of the reciprocals of the product of the eigenvalues taken three at a time of the following matrix:

$$\begin{bmatrix} 1 & 3 & 1 & 4 \\ 3 & 9 & 5 & 15 \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 2 & 3 \end{bmatrix}$$

A. $\dfrac{2}{7}$ B. $\dfrac{7}{2}$ C. $-\dfrac{2}{7}$ D. $-\dfrac{7}{2}$ E. NOTA

25. Suppose $f(x) = x^{2022} + 2x^{2021} + \cdots + 2022x + 2023$. The value of $f(z)f(z^2)f(z^3)\cdots f(z^{2023})$ when $z = \text{cis}(\frac{\pi}{1012})$ can be written in the form m^n. What is $m - n$?

A. 0 B. 2 C. 4 D. 6 E. NOTA

26. Let $\mathcal{R}$ region in the complex plane containing all complex numbers z with the property that there exists real numbers $a, b \in [0, 1]$ such that $z^2 + az + b = 0$. What is the area of $\mathcal{R}$?

A. $\frac{\pi}{6} + \frac{\sqrt{3}}{4}$ B. $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ C. $\frac{\pi}{12} + \frac{\sqrt{3}}{8}$ D. $\frac{\pi}{12} - \frac{\sqrt{3}}{8}$ E. NOTA

27. Given that

$$\sum_{k=0}^{\infty} \frac{\sin\left(\frac{k\pi}{6}\right)}{3^k} = \frac{a+b\sqrt{c}}{d}$$

where a, b, c and d are positive integers such that a, b and d share no prime factor and c is not divisible by the square of a prime, find $a+b+c+d$.

A. 172 B. 188 C. 194 D. 208 E. NOTA

28. The conic $3x^2+7xy+y^2+4x+5y+2=0$ is rotated counterclockwise by an angle $\theta \in (0, \frac{\pi}{2})$ so that it no longer has an xy term. This new rotated conic is given by the equation $Ax^2+Cy^2+Dx+Ey+F=0$. Find $A^2+C^2+D^2+E^2+F^2$ *(Hint: how do these values change under* ***any*** *such rotation?)*

A. $\frac{69}{2}$ B. $\frac{79}{2}$ C. $\frac{119}{2}$ D. $\frac{159}{2}$ E. NOTA

29. Which of the following is equivalent to the value of

$$\sin\left(\frac{\pi}{2022}\right) \cdot \sin\left(\frac{2\pi}{2022}\right) \cdot \sin\left(\frac{3\pi}{2022}\right) \cdots \sin\left(\frac{2020\pi}{2022}\right) \cdot \sin\left(\frac{2021\pi}{2022}\right)$$

A. $\frac{1011}{2^{2020}}$ B. $\frac{1011}{2^{2021}}$ C. $\frac{1}{2^{2021}}$ D. $\frac{1}{2^{2022}}$ E. NOTA

30. Allen has too much time on his hands! He wastes his time by drawing a 2021-gon in the xy-plane with none of its sides being vertical. Let $m_1, m_2, \cdots, m_n$ be the slopes of the n sides, listed in a counterclockwise order. Determine the value of the following sum.

$$S_n = m_1m_2 + m_2m_3 + \cdots m_{n-1}m_n + m_nm_1$$

A. -4042 B. -2022 C. -2021 D. -1011 E. NOTA