Student Delegates Theta Test

Written by Ritvik Teegavarapu

November 28, 2020

NOTA stands for "None of the Above." Good luck!

1 Algebra

1. Let a_k be an arithmetic progression with $a_1 > 0$ and $5a_{13} = 6a_{19}$. What is the smallest integer k such that $a_k < 0$?
A) 37
B) 42
C) 47
D) 50
E) NOTA

2. The function $f(x) = ba^{-x} + c$ has horizontal asymptote $y = 32$, y-intercept of 212 and passes through $(2, 112)$. Determine the value of $a \cdot c + b$.
A) 238
B) 88
C) 152
D) 302
E) NOTA

3. Let $x, y, z \in \mathbb{R}$ and the following equation be true.

$$7x^2 + 7y^2 + 7z^2 + 9xyz = 12$$

The minimum value of $x^2 + y^2 + z^2$, where x, y, z are real numbers, can be expressed as $\frac{a}{b}$, where a and b are relatively prime and $a, b \in \mathbb{Z}$. Determine $a + b$.
A) 8
B) 6
C) 4
D) 9
E) NOTA

4. Suppose x and y are positive reals satisfying the equation below.

$$\sqrt{xy} = x - y = \frac{1}{x+y} = k$$

The value of k is expressed as $a^{\frac{b}{c}}$, where a, b, and c are integers, b is less than c, and $\gcd(b, c) = 1$. Determine $a + b + c$.
A) 8
B) 9
C) 10
D) 11
E) NOTA

5. Consider the sequence $f(1) = 1$, $f(2) = \frac{1}{2}$, $f(3) = \frac{1+3}{2}$, $f(4) = \frac{1+3}{2+4}$, and this pattern continues. What is the minimum value of n, with $n > 1$, such that

$$|f(n) - 1| \leq \frac{1}{10}$$

A) 16
B) 18
C) 20
D) 22
E) NOTA

6. Let us define $f(x)$ as follows:

$$f(x) = \frac{2^{19}x + 2^{20}}{x^2 + 2^{20}x + 2^{20}}$$

Determine the value of the following sum.

$$f(1) + f(2) + f(4) + f(8) + \cdots + f(2^{20})$$

A) 8.5
B) 9.5
C) 10.5
D) 11.5
E) NOTA

7. Suppose that x, y, z are real, positive numbers satisfying

$$x^2 + xy + y^2 = 64$$

$$y^2 + yz + z^2 = 49$$

$$z^2 + xz + x^2 = 57$$

The value of $\sqrt[3]{xyz}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime. Determine $m + n$.
A) 56
B) 61
C) 67
D) 69
E) NOTA

2 Geometry

8. Right triangular prism $ABCDEF$ has triangular faces $\triangle ABC$ and $\triangle DEF$ and edges $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ has $\angle CBA = 90°$ and $\angle EAB = \angle CAB = 60°$. Given $AE = 2$, the volume of $ABCDEF$ can be expressed as $\frac{m}{n}$, where the fraction cannot be reduced, Determine $m + n$.
A) 15
B) 11
C) 8
D) 5
E) NOTA

9. Two cubes X and Y have different side lengths, such that the volume of cube X is numerically equal to the surface area of cube Y. If the surface area of cube X is numerically equal to six times the side length of cube Y, what is the ratio of the surface area of cube X to the volume of cube Y?
A) 1/1296
B) 1/216
C) 1/108
D) 1/36
E) NOTA

10. Suppose that $\triangle ABC$ has sidelengths $AB = 20$ and $AC = 17$. Let X be a point inside $\triangle ABC$ such that BX is perpendicular to CX, and AX is perpendicular to BC. If $|BX^4 - CX^4| = 2017$, and the length of BC can be expressed as $\sqrt{\frac{a}{b}}$, where a and b are relatively prime positive integers, determine $a + b$.
A) 2018
B) 2067
C) 2104
D) 2154
E) NOTA

11. Let ABCD be a convex quadrilateral with $AB = BC = 2, AD = 4$, and $\angle ABC = 120°$. Let M be the midpoint of BD. The $\angle AMC = 90°$, and the length of segment CD can be expressed as $a\sqrt{b}$. Determine $a + b$.
A) 8
B) 9
C) 10
D) 11
E) NOTA

12. Let ABCD be a cyclic quadrilateral where AB = 4, BC = 11, CD = 8, and DA = 5. If BC and DA intersect at X, and if the area of $\triangle$ XAB can be expressed as $a\sqrt{b}$, where a and b are relatively prime, and b is not divisible by a perfect square, find $a \cdot b$.
A) 15
B) 18
C) 24
D) 30
E) NOTA

13. Let ABE be a triangle with $AB/3 = BE/4 = EA/5$. Let D $\neq$ A be on line AE such that $AE = ED$ and D is closer to E than to A. Moreover, let C be a point such that $BCDE$ is a parallelogram. Furthermore, let M be on line CD such that AM bisects $\angle$ BAE, and let P be the intersection of AM and BE. If the ratio of the length PM to the perimeter of triangle ABE can be expressed as $\frac{\sqrt{a}}{b}$, determine $a + b$.
A) 11
B) 12
C) 13
D) 14
E) NOTA

14. Let $ABCD$ be a parallelogram with $BC = 2020$. Let T be the midpoint of BC and let N be the point such that $DANT$ is a parallelogram. What is length of NC?
A) 1010
B) 2020
C) 3030
D) 4040
E) NOTA

15. Right triangle with right angle B and integer side lengths has BD as the altitude. E and F are the incenters of triangles $\triangle ADB$ and $\triangle BDC$ respectively. Line EF is extended and intersects BC at G, and AB at H. If $AB = 15$ and $BC = 8$, and if the area of triangle $\triangle BGH$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers, determine $a + b$.
A) 7489
B) 7598
C) 7865
D) 7984
E) NOTA

3 Number Theory

16. Given that $\frac{1}{31} = 0.\overline{a_1 a_2 \cdots a_n}$ (where $a_1, a_2, \cdots a_n$ are the digits of the decimal expansion of $\frac{1}{31}$), what is the minimum value of n?
A) 12
B) 15
C) 18
D) 21
E) NOTA

17. Say we pick positive integers p and q such that the following equation is held.

$$\frac{p + \frac{1}{q}}{q + \frac{1}{p}} = 20$$

What is the largest integer n we can pick such that $\frac{p+q}{n}$ will always remain an integer?
A) 15
B) 16
C) 17
D) 18
E) NOTA

18. Let $S = \{1, 2, 3, 4, \cdots, 10\}$. In how many ways can two (not necessarily distinct) elements a, b be taken from S such that $\frac{a}{b}$ is in lowest terms, or that $\gcd(a, b) = 1$ is met?
A) 37
B) 43
C) 54
D) 63
E) NOTA

19. An `immovable` number is a integer which can be expressed as $x \cdot \lfloor x \rfloor \cdot \lceil x \rceil$ where x is a real number. Determine the number of `immovable` numbers that are positive and less than or equal to 1000.
A) 278
B) 296
C) 305
D) 331
E) NOTA

20. Let S be the set of all rational numbers $x \in [0, 1]$ with repeating base 6 expansion $x = 0.\overline{a_1 a_2 \cdots a_6}$. x can also be written as $0.\overline{a_1 a_2 \cdots a_k}$ for some finite sequence $\{a_i\}_{i=1}^{k}$ of distinct non-negative integers less than 6. What is the sum of all numbers that can be written in this form? (Put answer in base 10)
A) 678
B) 764
C) 864
D) 978
E) NOTA

21. If the following summation can be expressed as the irreducible fraction $\frac{a}{b}$, determine $a + b$. The function $\phi(n)$ denotes the number of positive integers less than or equal to n that are relatively prime to n.

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{101^n - 1}$$

A) 10201
B) 10101
C) 10010
D) 10001
E) NOTA

22. How many positive integers are factors of 12! but not $(7!)^2$?
A) 486
B) 507
C) 522
D) 540
E) NOTA

23. Denote the set $P(a, b)$ as the set of integers p such that p can be represented in the form $m \cdot a + n \cdot b$, for some non-negative integers m and n. Determine the sum of all positive values of x such that $S(18, 32)$ is a subset of $S(3, x)$.
A) 198
B) 207
C) 218
D) 235
E) NOTA

4 Combinatorics

24. Let $\{a_i\}$ for $1 \leq i \leq 10$ be a finite sequence of 10 integers such that for all odd i, $a_i = 1$ or -1, and for all even i, $a_i = -1, 1$, or 0. How many sequences $\{a_i\}$ exist such that $a_1 + a_2 + \cdots a_{10} = 0$?
A) 1004
B) 1056
C) 1086
D) 1104
E) NOTA

25. Derek is inside his house, consisting of points a, b such that $0 \leq a, b, \leq 2020$. If he is currently on the point (x, y), he can move to either $(x, y+1)$, $(x, y-1)$, or $(x+1, y)$. Given that he cannot revisit any point (x, y) that he has already visited before, and he cannot travel to any point outside of his house if the number of paths Derek can take to get from $(2020, 2020)$ to $(0, 0)$ can be expressed as 2020^b for integer b, determine b.
A) 2018
B) 2019
C) 2020
D) 2021
E) NOTA

26. A finite set of distinct, non negative integers $\{a_1, a_2, \cdots, a_k\}$ is `litty` if the integer function $f(n) = (n+a_1)(n+a_2)\cdots(n+a_k)$ has no common divisors over all terms. Alternatively, we have that $\gcd(f(1), f(2), \cdots, f(n)) = 1$. How many `litty` sets only have members less than 10?
A) 32
B) 44
C) 56
D) 70
E) NOTA

27. There are 2020 people labeled $1, 2, \cdots, 2020$ working together on 2020 tasks with people $1, \cdots, i$ working on the i^{th} task. There is one imposter among the people. If none of the people working on a task are the imposter, then the task will be successful. However, if the imposter is working on a task, there is a $\frac{1}{2}$ chance that it will be a failure. Given that the first 2019 tasks succeed, and the 2020th task has failed (we'll get 'em next time), the probability the 2020th person is the imposter can be expressed as P.

$$P = \frac{2^b}{2^c - d}$$

If P can be expressed in the above form for some integers b, c, and d, determine $b + c + d$.
A) 4019
B) 4028
C) 4040
D) 4043
E) NOTA

28. Let $f(n)$ have a parameter n which is a non-negative integer, and return an integer between 0 and $n-1$ at random (note that $f(0) = 0$ always). Let P be the the expected value of $f(f(2020))$. If the fractional part of P (which can be expressed as $P - \lfloor P \rfloor$) can be expressed as an irreducible fraction $\frac{b}{c}$ for integers b and c, where $\gcd(b, c) = 1$, determine $b + c$.
A) 5051
B) 5065
C) 5078
D) 5089
E) NOTA

29. Let $a_1, a_2, \cdots, a_n$ be a sequence of real numbers. Call a k-inversion ($0 < k \leq n$) be a sequence of indices $i_1, i_2, i_3, \cdots, i_k$ such that $i_1 < i_2 < \cdots < i_k$ but $a_{i_1} > a_{i_2} \cdots > a_{i_k}$. If the expected number of 5-inversions in a random permutation of the set of $\{1, 2, 3, \cdots, 12\}$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b what is $a + b$?
A) 36
B) 34
C) 32
D) 30
E) NOTA

30. Deeksha is bored, and has a coin. She flips this coin until there is a head followed by two tails. If the probability that this will take exactly 12 flips can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, determine $a+b$.
A) 4198
B) 4239
C) 4324
D) 4453
E) NOTA