Student Delegates Mu Test

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NOTA stands for "None of the Above." Good luck!

1 Limits

1. Evaluate the following limit

$$\lim_{n\to\infty} \frac{\sqrt{n^2-1^2}+\sqrt{n^2-2^2}+\cdots+\sqrt{n^2-(n-1)^2}}{n^2}$$

A) $\pi/4$
B) $\pi/2$
C) $3\pi/2$
D) 2π
E) NOTA

2. If the following limit can be expressed as e^A, what is the value of A?

$$\lim_{x\to 0} \left(\frac{\sinh(x)}{x}\right)^{\frac{1}{x^2}}$$

A) 2
B) 1
C) $\frac{1}{4}$
D) $\frac{1}{8}$
E) NOTA

3. Evaluate the following limit.

$$\lim_{n\to\infty} \frac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+1/2}}$$

A) $4\sqrt{\pi e}$
B) $2\sqrt{\pi e}$
C) $\sqrt{2\pi e}$
D) $\sqrt{\pi e}$
E) NOTA

4. For a given $x > 0$, let a_n be the sequence defined by $a_1 = x$ for $n = 1$ and $a_n = x^{a_{n-1}}$ for $n \geq 2$. Let x is the largest value for which $\lim_{n\to\infty} a_n$ converges. Determine the value of x.
A) $e^{\frac{2}{e}}$
B) $e^{\frac{1}{e}}$
C) $e^{\frac{1}{2e}}$
D) 1
E) NOTA

5. Evaluate the following limit.

$$\lim_{n\to\infty} \left(\tan\left(\frac{\pi}{2n}\right) \tan\left(\frac{\pi}{2n}\right) \cdots \tan\left(\frac{n\pi}{2n}\right) \right)^{\frac{1}{n}}$$

A) 3
B) $\frac{2}{3}$
C) 2
D) 1
E) NOTA

6. Evaluate the following limit.

$$\lim_{n\to\infty} \frac{1}{n} \left(\prod_{k=0}^{n} (n+k) \right)^{\frac{1}{n}}$$

A) $7/e$
B) $4/e$
C) $8/e$
D) $2/e$
E) NOTA

7. Determine the value of the following limit at $p = 2020$.

$$\lim_{n\to\infty} \frac{1^p + 2^p + 3^p + \cdots n^p}{n^{p+1}}$$

A) 1/2021
B) 1/2022
C) 1/2023
D) 1/2024
E) NOTA

2 Derivatives

8. Odin is standing at the point $(1, 0)$ waiting for the school bus to pick him up. However, his bus driver is sick, and Mrs. Frizzle (with the Frizz, no way!). Odin hopes this will be a normal field trip, but lo and behold, it is not. Mrs. Frizzle drives the Magic School Bus at a speed of 1 meter per second along the graph $x = \frac{2}{5}y^2$. With this, help Odin find the rate, in radians per second, at which Odin's head is turning clockwise when Mrs. Frizzle passes $x = 1$.
A) $2/\sqrt{35}$
B) $3/\sqrt{57}$
C) $4/\sqrt{65}$
D) $5/\sqrt{85}$
E) NOTA

9. If

$$f(x) = -100 + \sum_{k=1}^{100} x^k$$

find the largest prime divisor of $f'(1)$.
A) 83
B) 89
C) 101
D) 103
E) NOTA

10. What is the minimum vertical distance between the graphs of $y = 2+\cos(x)$ and $y = \sin(x)$?
A) $2 + \sqrt{2}$
B) $2 - 2\sqrt{2}$
C) $1 + \sqrt{2}$
D) $1 - 2\sqrt{2}$
E) NOTA

11. The graph of the cubic $y = x^3 - 9x^2 + 24x + 4$ has a local minimum at (x_1, y_1) and local maximum at (x_2, y_2). Determine the determinant of the following matrix.

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

A) 42
B) 56
C) 68
D) 84
E) NOTA

12. Let $z = \frac{1}{2}(\sqrt{2} + i\sqrt{2})$. If the following sum can be expressed as $a - bi$, find $a + b$.

$$\sum_{k=0}^{13} \frac{1}{1 - ze^{\frac{k\pi i}{7}}}$$

A) 11
B) 12
C) 13
D) 14
E) NOTA

13. Let $f \in \mathbb{R}_{>0} \to \mathbb{R}$ (where $\mathbb{R}_{>0}$ is the set of all real positive numbers) be differentiable and satisfy the equation

$$f(y) - f(x) = \frac{x^x}{y^y} f\left(\frac{y^y}{x^x}\right)$$

for all real $x, y > 0$. Furthermore, $f'(1) = 1$. Compute $\frac{f'(2020^2)}{f'(2020)}$.
A) 1010
B) 2020
C) 4040
D) 8080
E) NOTA

14. The equation of $e^{2x} = k\sqrt{x}$ has exactly one real solution x for some constant k. Find the greatest integer less than or equal to $100k^2$.
A) 1087
B) 997
C) 925
D) 908
E) NOTA

3 Integrals

15. Evaluate the below expression.

$$\frac{\displaystyle\int_0^1 \text{arccot}(1-x+x^2)\ \text{dx}}{\displaystyle\int_0^1 \arctan(x)\ \text{dx}}$$

A) 1/8
B) 1/4
C) 1/2
D) 1
E) NOTA

16. If the following integral can be expressed as $ae^b + c$, determine the value of $a + b + c$.

$$\int_0^{2025} e^{\sqrt{x}}\ dx$$

A) 127
B) 135
C) 145
D) 152
E) NOTA

17. If the following integral is of the form $\frac{a\pi}{b}$, with a and b relatively prime, determine $a + b$.

$$\int_0^{\pi} \sin^{10}(x)\ \text{dx}$$

A) 78
B) 146
C) 245
D) 319
E) NOTA

18. If the following limit evaluates to the rational number $\frac{a}{b}$, in which the fraction is irreducible, determine $a+b$.

$$\lim_{n\to\infty} n^2 \int_0^{\frac{1}{n}} x^{x+1}\ dx$$

A) 2
B) 3
C) 4
D) 5
E) NOTA

19. What is the sum of all integers m for $1 \leq m \leq 10$ such that the following equation is true?

$$\int_0^{\pi} (\cos(x))(\cos(2x))\cdots(\cos(mx))\ \mathrm{dx} = 0$$

A) 25
B) 29
C) 33
D) 37
E) NOTA

20. Determine what the following integral evaluates to.

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3(2x)}{\ln(\csc(x))}\ \mathrm{dx}$$

A) $8\ln(7/2)$
B) $8\ln(5/2)$
C) $8\ln(3/2)$
D) $8\ln(1/2)$
E) NOTA

21. If the following integral can be expressed in the form $\frac{A}{\pi}$, determine the value of A.

$$\int_0^{\frac{\pi}{2}} \frac{x^2+2020(2019)}{(x\sin(x)+2020\cos(x))^2}\ \mathrm{dx}$$

A) 505
B) 1010
C) 1515
D) 2020
E) NOTA

22. If the following integral can be expressed as $\frac{\pi^a - b}{c}$, find $a + b + c$. Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x; $\{x\}$ is the fractional part of x, or $x - \lfloor x \rfloor$

$$\int_1^\infty \frac{2x\{x\} - \{x\}^2}{x^2 \lfloor x \rfloor^2} \text{ dx}$$

A) 14
B) 15
C) 16
D) 17
E) NOTA

23. If the following expression can be expressed as $\frac{a}{b}$, with the fraction being irreducible, determine $a + b$.

$$\frac{\displaystyle\int_0^\infty (1 + x^2)^{-2020} \, dx}{\displaystyle\int_0^\infty (1 + x^2)^{-2019} \, dx}$$

A) 8075
B) 8097
C) 8124
D) 8156
E) NOTA

24. The value of the integral can be expressed as $2\pi \ln(b)$. Determine b.

$$\int_0^\pi \ln(1 - 4040\cos(x) + 2020^2) \, dx$$

A) 2018
B) 2020
C) 2022
D) 2024
E) NOTA

4 Sequences and Series

25. To start off, let's do some Fibonacci! If the following sum evaluates to the value $\frac{a}{b}$, where F_n denotes the Fibonacci sequence, determine $a+b$, with $F_1 = 1$, and $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$.

$$S = \sum_{n=1}^{\infty} \frac{F_n}{9^n}$$

A) 68
B) 72
C) 74
D) 80
E) NOTA

26. The summation

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty} \frac{1}{a^2b + 2ab + ab^2}$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime integers.
A) 5/4
B) 3/2
C) 7/4
D) 9/4
E) NOTA

27. If the following sum evaluates to $a - \pi^b$, with a and b being positive integers, determine $a + b$.

$$\sum_{n=1}^{\infty} \frac{1}{(n^2 + n)^3}$$

A) 8
B) 9
C) 10
D) 11
E) NOTA

28. If we define $f(x)$ as the following, and if $f\left(\frac{\pi}{3}\right)$ can be expressed as $e^a \sin(b)$, determine $a+b$.

$$f(x) = \sum_{n=0}^{\infty} \frac{\sin(nx)}{n!}$$

A) $(1+\sqrt{3})/2$
B) $1+\sqrt{3}/2$
C) $(1+\sqrt{6})/2$
D) $1+\sqrt{6}$
E) NOTA

29. Evaluate the following integral, where $\{x\}$ is the fractional part of x, which we will define as $x - \lfloor x \rfloor$.

$$\int_0^1 \left\{ \frac{(-1)^{\lfloor \frac{1}{x} \rfloor}}{x} \right\} \mathrm{dx}$$

A) $1 - \ln(\pi/2)$
B) $1 - \ln(\pi/4)$
C) $2 - \ln(\pi/2)$
D) $2 - \ln(\pi/4)$
E) NOTA

30. Evaluate the following integral.

$$\int_0^1 \frac{(1-x)}{\ln(x)} \left(x + x^2 + x^{2^2} + \cdots \right) \mathrm{dx}$$

A) $2\ln(2)$
B) $-\ln(5)$
C) $-2\ln(3)$
D) $\ln(5)$
E) NOTA