Student Delegates Alpha Test

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NOTA stands for "None of the Above." Good luck!

1 Trigonometry

1. Determine the value of the below expression.

$$\frac{1}{\sin^2\left(\frac{\pi}{10}\right)}+\frac{1}{\sin^2\left(\frac{3\pi}{10}\right)}$$

A) 6
B) 12
C) 18
D) 24
E) NOTA

2. Determine the number of solutions for the following equation, where $0^\circ \leq x \leq 2020^\circ$.

$$\sin^{10}(x)+\cos^{10}(x)=\frac{29}{16}\cos^4(2x)$$

A) 41
B) 42
C) 43
D) 44
E) NOTA

3. If the 2020-th smallest x, with $x>1$, that satisfies the following relation, can be expressed as $e^{\frac{a\pi}{b}}$ for some relatively prime positive integers a and b, determine $a+b$.

$$\sin(\ln(x))+2\cos(3\ln(x))\sin(2\ln(x))=0$$

A) 2023
B)2024
C) 2025
D) 2026
E) NOTA

4. Find the sum of all solutions α (in degrees) such that $0° < \alpha < 90°$ and the following equation holds true.

$$1+\sqrt{3}\tan(60° - \alpha°) = \frac{1}{\sin(\alpha°)}$$

A) 80
B) 90
C) 100
D) 110
E) NOTA

5. Determine the value of the following expression.

$$\frac{\tan 1°}{1+\tan(1°)} + \frac{\tan 2°}{1+\tan(2°)} + \cdots \frac{\tan 89°}{1+\tan(89°)}$$

A) 85/2
B) 43
C) 87/2
D) 44
E) NOTA

6. Pick a and b, which are non-negative real numbers such that $\sin(ax+b) = \sin(2020x)$. If the least possible value of a can be expressed as $a\pi - b$, determine $a+b$.
A) 2543
B) 2567
C) 2589
D) 2603
E) NOTA

7. Determine the value of the following expression.

$$\frac{1}{1-\cos\left(\frac{\pi}{9}\right)} + \frac{1}{1-\cos\left(\frac{5\pi}{9}\right)} + \frac{1}{1-\cos\left(\frac{7\pi}{9}\right)}$$

A) 17
B) 18
C) 19
D) 20
E) NOTA

8. The function $\zeta(x)$ is of the following form:

$$\zeta(x) = \prod_{i=1}^{n} \zeta_i(a_i x)$$

where a_i is a real number and $\zeta_i(x)$ is either $\sin(x)$ or $\cos(x)$ for $i = 1, 2, \cdots, n$. Additionally, $\zeta(x)$ is known to have zeros at every integer between 1 and 2020 (inclusive) except for one integer c. Determine the sum of all possible values of c.
A) 2020
B) 2047
C) 2063
D) 2099
E) NOTA

9. Let $0 < \theta < 2\pi$ be a real number for which

$$\cos(\theta) + \cos(2\theta) + \cos(3\theta) + \cdots \cos(2020\theta) = 0$$

for some $\theta = \frac{\pi}{n}$ for some positive integer n. Compute the sum of the possible values of $n \leq 2020$.
A) 1926
B) 1876
C) 1984
D) 1934
E) NOTA

10. Let $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers. Consider a continuous function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ which satisfies

$$f(x^2) + f(y^2) = f\left(\frac{x^2y^2 - 2xy + 1}{x^2 + 2xy + y^2}\right)$$

where x, y are positive real numbers with $xy > 1$. Given that $f(0) = 2020$ and $f(1) = 1010$, determine the value of $f(3)$.
A) 2020/3
B) 2020/5
C) 2020/7
D) 2020/9
E) NOTA

2 Complex and Polar

11. How many different values of a complex number z are there which satisfy $\text{Im}(z) = z^2 - z$ (where $\text{Im}(z)$ denotes the imaginary part of z)?
A) 0
B) 2
C) 4
D) 5
E) NOTA

12. Let $f(x)$ be a complex monic quadratic with roots $\frac{1}{3}$, $\frac{2}{3}$. The polynomial $f(X)$ is of the form $X^2 + bX + c$, where b, c, and X are complex numbers). If $|z| = 1$, what is the sum of all possible values of $f(z)$ such that $f(z) = \overline{f(z)}$?
A) 5/3
B) 7/3
C) 8/3
D) 11/3
E) NOTA

13. Given a complex number z such that $z^{13} = 1$, find the sum of all possible values of $z + z^3 + z^4 + z^9 + z^{10} + z^{12}$.
A) 3
B) 4
C) 5
D) 6
E) NOTA

14. Compute the number of ordered pairs of positive integers (m, n) with $m + n \leq 64$ such that there exists at least one complex number z such that $|z| = 1$ and $z^m + z^n + \sqrt{2} = 0$.
A) 150
B) 200
C) 250
D) 300
E) NOTA

15. Let a, b, c be real angles such that

$$3\sin(a) + 4\sin(b) + 5\sin(c) = 0$$

$$3\cos(a) + 4\cos(b) + 5\cos(c) = 0$$

The maximum value of the expression $\frac{\sin(b)\sin(c)}{\sin^2(a)}$ can be expressed as $\frac{a}{b}$ for relatively prime a and b. Determine $a + b$.
A) 87
B) 89
C) 93
D) 97
E) NOTA

16. Solve for x in the following equation.

$$e^{\arcsin(x) \cdot i} = 1$$

A) 0
B) 1/2
C) $\frac{\sqrt{3}}{2}$
D) $-1/2$
E) NOTA

17. Let S be the set of complex numbers of the form $c + di$ such that $c + di = (a + bi)^{12}$ for some integers a and b. Find the largest integer that must divide d for all numbers in S.
A) 576
B) 624
C) 648
D) 792
E) NOTA

18. Suppose that the complex number z satisfies $|z| = \left|z^2 + 1\right|$. K is the maximum possible value of $|z|$. K^4 can be expressed in the form $\dfrac{r + \sqrt{s}}{t}$ for integers r, s, and t, where s is not divisible by a perfect square. Find $r + s + t$.
A) 107
B) 89
C) 76
D) 54
E) NOTA

19. Ainsley the Artist initially paints every complex number black. When Ainsley toggles a complex number, she paints it white if it was previously black, and black if it was previously white. For each $k = 1, 2, \ldots, 20$, Ainsley progressively toggles the roots of $x^{2k} + x^k + 1$. Let N be the number of complex numbers are white at the end of this process. Find N.
A) 180
B) 200
C) 220
D) 240
E) NOTA

20. Let P be the set of $\frac{m}{n}$ where every element of P is a root of $x^{97} - 1 = 0$ and

$$m = e^{\frac{2\pi i}{p}}, n = e^{\frac{2\pi i}{q}}$$

for integers (p, q). Determine the number of ordered pairs (p, q).
A) 100
B) 120
C) 1400
D) 160
E) NOTA

3 Matrices and Vectors

21. Find the unit vector in the direction of $< 3, 4, 12 >$.
A) $< 1/4, 1/3, 1 >$
B) $< 3/13, 4/13, 5/13 >$
C) $< 1, 1, 1 >$
D) $< 1, 4/3, 4 >$
E) NOTA

22. What is the magnitude of the vector $< 3, 4, 5 >$?
A) 3
B) 4
C) 5
D) $5\sqrt{2}$
E) NOTA

23. Given that a, b, and c are real numbers satisfying $a+b+c \geq 1010$, find the minimum value of the expression below.

$$\sqrt{a^2+ab+2b^2}+\sqrt{b^2+bc+2c^2}+\sqrt{c^2+ac+2a^2}$$

A) 505
B) 1010
C) 2020
D) 4040
E) NOTA

24. Evaluate the following determinant, with all of the arguments of cos being in radians.

$$\left\|\begin{matrix}\cos(1) & \cos(2) & \cos(3)\\ \cos(4) & \cos(5) & \cos(6)\\ \cos(7) & \cos(8) & \cos(9)\end{matrix}\right\|$$

A) $\sqrt{3}$
B) $\sqrt{2}$
C) $\sqrt{2}/2$
D) $1/2$
E) NOTA

25. A *permutation matrix* is a square matrix that has exactly one 1 in each row and column and 0s everywhere else. Given a permutation matrix A, define the *period* of A to be the smallest positive integer k such that $A^n = A^{n+k}$ for all positive integers n (such a period always exists). What is the largest possible period of a 20×20 permutation matrix?
A) 420
B) 480
C) 540
D) 600
E) NOTA

26. Let A be a 20×20 matrix, with entries $a_{i,j} = \gcd(i, j)$ for $1 \leq i, j \leq 16$. Determine the number of factors of $\det(A)$.
A) 448
B) 486
C) 528
D) 586
E) NOTA

27. A p-matrix is a structured, infinite, collection of numbers starting with 1 as the first entry in the first row. In general, the first entry of each row of is $\frac{1}{2p}$ times the first entry of the previous row. Then, each succeeding term in a row is $\frac{1}{p}$ times the previous term in the same row. For example, a 3-matrix is constructed as follows:

$$\begin{array}{cccccc} 1 & \dfrac{1}{3} & \dfrac{1}{9} & \dfrac{1}{27} & \cdots \\ & \dfrac{1}{6} & \dfrac{1}{18} & \dfrac{1}{54} & \cdots \\ & & \dfrac{1}{36} & \dfrac{1}{108} & \cdots \\ & & & \dfrac{1}{216} & \cdots \\ & & & & \ddots \end{array}$$

If the sum of all the terms in a 2020-matrix can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers, find the remainder when $m+n$ is divided by 2020.
A) 5
B) 4
C) 1
D) 0
E) NOTA

28. A magic 3×5 board can toggle its cells between black and white. Define a `arrangement` to be an assignment of black or white to each of the board's 15 cells (so there are 2^{15} `arrangements` total). Every day after Day 1, at the beginning of the day, the board gets bored with its black-white `arrangement` and makes a new one. However, the board always wants to be unique and will die if any two of its `arrangements` are less than 3 cells different from each other. Furthermore, the board dies if it becomes all white. If the board begins with all cells black on Day 1, compute the maximum number of days it can stay alive.
A) 2005
B) 2016
C) 2032
D) 2047
E) NOTA

29. Let $\mathbb{R}^n$ be the set of vectors $(x_1, x_2, \cdots, x_n)$, where $x_1, x_2, \cdots, x_n$ are all real numbers. Let $||(x_1, x_2, \cdots, x_n)|| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$. Now let S be the set in $\mathbb{R}^9$ given by

$$S = \{(x, y, z) : x, y, z \in \mathbb{R}^3, 1 = ||x|| = ||y - x|| = ||z - y||\}$$

If a point (x, y, z) is chose uniformly at random from S, what is the expected value of $||z||^2$?
A) 3/2
B) 3
C) 5/2
D) 7/2
E) NOTA

30. Let n be a positive integer. Points A, B, C are selected from a unit n-dimensional hypercube.

$$\mathbb{H} = \{(x_1, x_2, \cdots, x_n) \mid 0 \leq x_i \leq 1 \text{ for } i = 1, 2, \cdots, n\}$$

Given that the maximum possible area of $\triangle ABC$ is a positive integer, minimize n.
A) 11
B) 10
C) 9
D) 8
E) NOTA